Z transform

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Z Transform

- Previously, we discussed the discrete-time Fourier transform (DTFT)
- Here, we will begin our discussion of the Z-transform (ZT)
  - ZT can be thought of as a generalization of the DTFT
  - ZT is more complex than DTFT (both literally and figuratively), but provides a great deal of insight into system design and behavior
  - Provide insight into the relationships between frequency using ZT and DTFT relationships

The Discrete-Time Fourier Transform (DTFT) and the *Z*-transform (ZT)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega}\,d\omega \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega}$$

- The first equation asserts that we can represent any time function *x[n]* by a linear combination of complex exponentials

$$e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$$

- The second equation tells us how to compute the complex weighting factors $X(e^{j\omega})$
- In going from the DTFT to the ZT we replace $e^{j\omega n}$ by z^n

Generalizing the frequency variable

- In going from the DTFT to the ZT we replace $e^{j\omega n}$ by z^n
- z^n can be thought of as a generalization of $e^{j\omega n}$
- For an arbitrary *z*, using polar notation we obtain $z=\rho e^{j\omega}$ so

$$z^n = \rho^n e^{j\omega n}$$

- If both ρ and ω are real, then z^n can be thought of as a complex exponential (*i.e.* sines and cosines) with a real temporal envelope that can be either exponentially decaying or expanding

Definition of the *Z*-transform

- Recall that the DTFT is

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega}$$

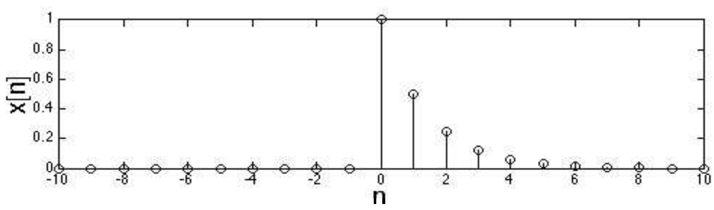
- Since we are replacing (generalizing) the complex exponential building blocks $e^{j\omega n}$ by z^n, a reasonable extension of $X(e^{j\omega})$ would be

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- Again, think of this as building up the time function by a weighted sums of functions z^n instead of $e^{j\omega n}$

Computing the Z-*transform: an example*

- Example 1: **Consider the time function** $x[n] = \alpha^n u[n]$



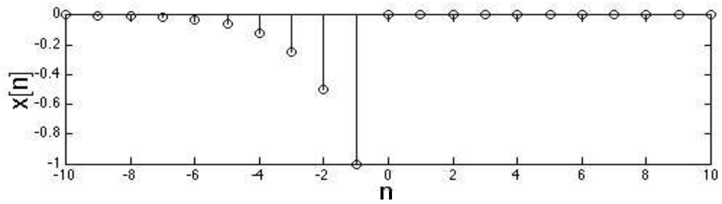
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

$$= \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}$$

Another example …

- Example 2: **Now consider the time function**

$$x[n] = -\alpha^n u[-n-1]$$



$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = \sum_{n=-\infty}^{-1} (\alpha z^{-1})^n$$

- Let $l = -n; n = -\infty \Rightarrow l = \infty; n = -1 \Rightarrow l = 1$
- Then, $\sum_{n=-\infty}^{-1} (\alpha z^{-1})^n = \sum_{l=1}^{\infty} -(z\alpha^{-1})^l = 1 - \sum_{l=0}^{\infty} (z\alpha^{-1})^l = 1 - \frac{1}{1 - z\alpha^{-1}} = \frac{1}{1 - \alpha z^{-1}}$

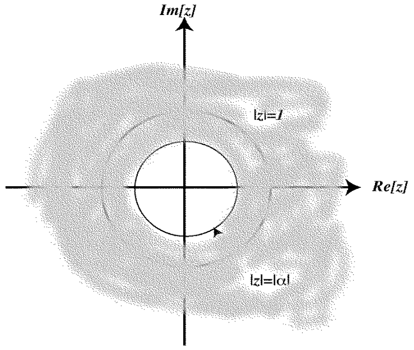
The importance of the region of convergence

- Did you notice that the *Z*-transforms were identical for Examples 1 and 2 even though the time functions were different? Yes, indeed, very different time functions can have the same *Z*-transform! What's missing in this characterization? The region of convergence (ROC).

- In Example 1, the sum $X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n}$ converges only for $|z| > |\alpha|$
- In Example 2, the sum $X(z) = \sum_{n=-\infty}^{-1} \alpha^n z^{-n}$ converges only for $|\alpha| > |z|$
- So in general, we must specify not only the *Z*-transform corresponding to a time function, but its ROC as well.

What shapes are ROCs for *Z*-transforms?

- In Example 1, the ROC was $|z|>|\alpha|$
- We can represent this graphically as:



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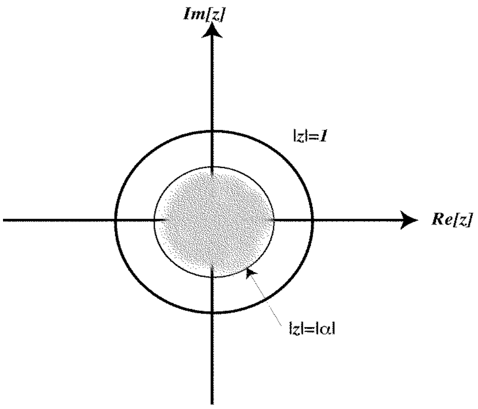
What shapes are ROCs for *Z*-transforms?

- In Example 2, the ROC was $|z|<|\alpha|$
- We can represent this graphically as:

(ROC is shaded area)



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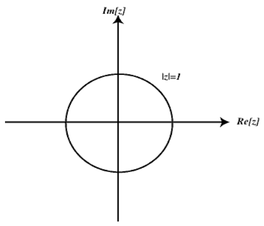
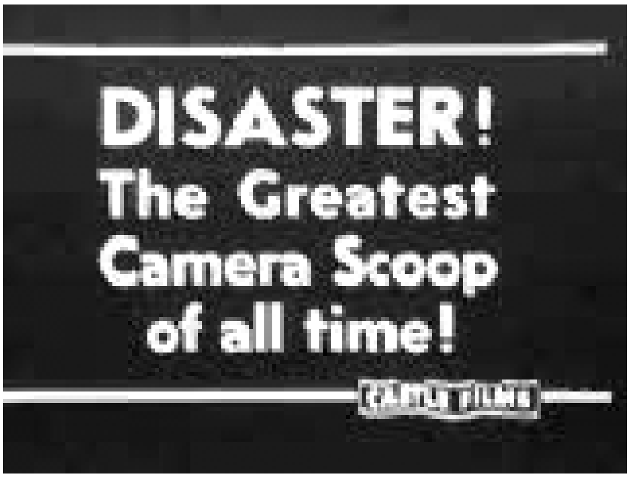
Stability and the ROC

- It can be shown that an LSI system is stable if the ROC includes the unit circle (UC), which is the locus of points for which $|z| = 1$



- Comment: this is exactly the same condition that is required for the DTFT to exist

Example



The inverse *Z*-transform

- Did you notice that we didn't talk about inverse *z*-transforms yet?
- It can be shown (see the text) that the inverse *z*-transform can be formally expressed as

$$x[n] = \frac{1}{2\pi j}\oint_c X(z)z^{n-1}dz$$

- Comments:
  - Unlike the DTFT, this integral is over a **complex variable,** *z* and we need complex residue calculus to evaluate it formally
  - The contour of integration, *c*, is a circle around the origin that lies inside the ROC
  - We will never need to actually evaluate this integral in this course

Summary on Z transform

- The *z*-transform is based on a generalization of the frequency representation used for the DTFT
- Different time functions may have the same *z*-transforms; the ROC is needed as well
- The ROC is bounded by one or more circles in the *z*-plane centered at its origin
- An LSI system is stable if the ROC includes the unit circle
- The inverse *z*-transform can only be evaluated using complex contour integration