Spectrum Analysis

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- Introduction
- Periodogram
- Correlogram
- Windowing and spectrum analysis

Spectrum analysis

Detection of signals (radar)

Signal modeling

Noise removal

Basic idea: *Determine the correlation between elements of a signal to enable modeling, compression. Determine power spectral content of the signal to enable noise removal, detection,...*

Correlation

- Signal Modeling
- Related to statistical description of signals (noise)
- Related to the Power Spectral Density (PSD)

$$R_{x_1x_1}(n) = x_1(n) \otimes x_1(n) = \sum_{m=-\infty}^{\infty} x_1(m)x_1(n+m)$$ Autocorrelation

↓ DFT

$$S_N(k) = \frac{1}{N} X(k)X^*(k) = \frac{1}{N}\left|X(k)\right|^2$$ Power spectral density

Spectrum analysis

Why spectrum analysis?

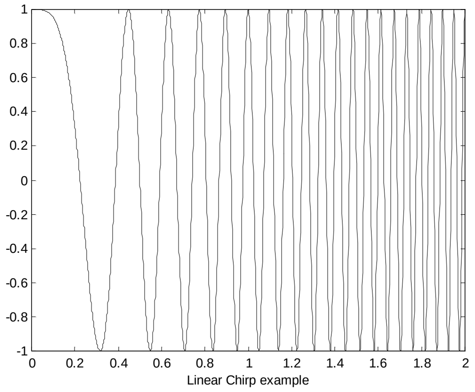
Filtering is a convolution in spectral domain

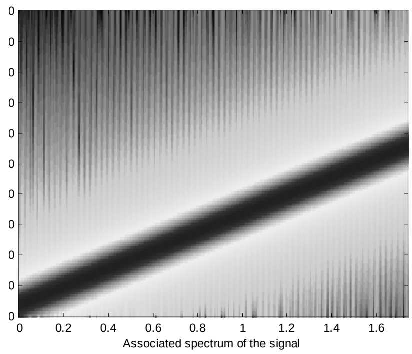
Information representation in the spectral domain can be:

- Easier to interpret
- Easier to visualize
- More concentrated
- More natural (audio frequencies)

Spectrum analysis

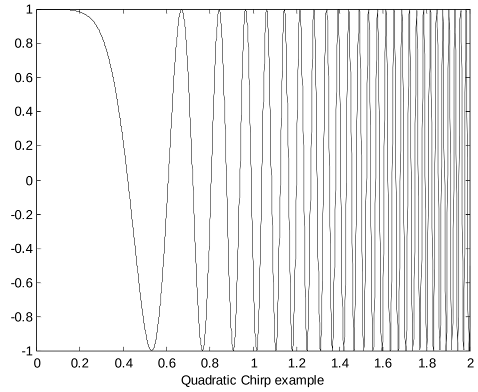
<u>Example</u>

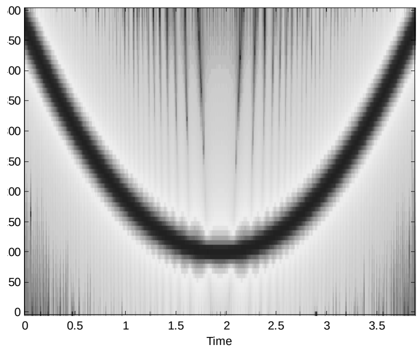




Spectrum analysis

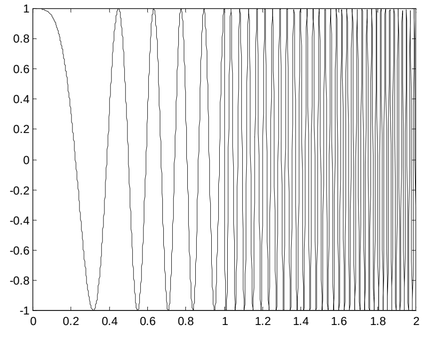
Example

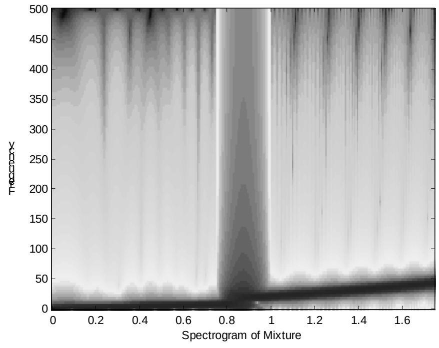




Spectrum analysis

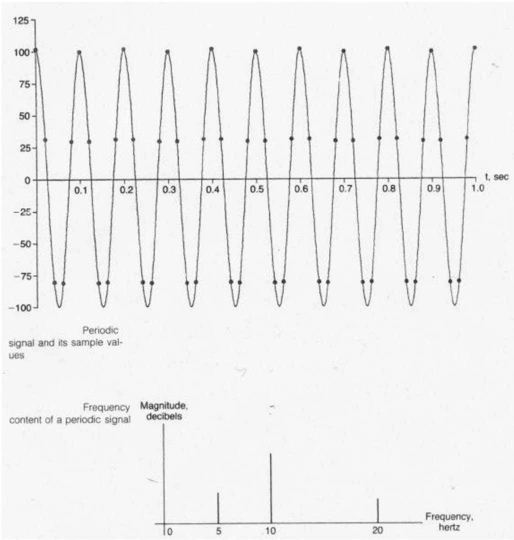
Example





Spectrum analysis

Example:



Periodogram based spectral estimation

Technique:

$$S_N(k) = \frac{1}{N}|X(k)|^2 = \frac{1}{N}X(k)X^*(k), \quad k = 0,1,\ldots,N-1$$

Calculated using DFT.

$x_1(n)$ → DFT → $|\,|^2$ → $S_N(k) = \frac{1}{N}|X(k)|^2$

Problem?

What about long signals (thousands of samples)?

Periodogram based spectral estimation

Solution:

x(n) is decomposed in M smaller overlapping sections.

The spectrum is calculated for each section

$$S_N(k) = \frac{1}{M}\sum_{m=1}^{M} S_{Nm}(k)$$

Assumes stationarity and ergodicity

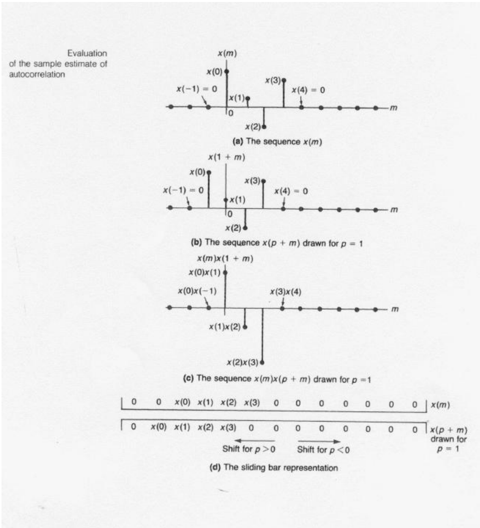
Correlogram

$$R_{x_1x1}(n) = x_1(n) \otimes x_1(n) = \sum_{m=-\infty}^{\infty} x_1(m)x_1(n+m)$$ Autocorrelation

$$R_N(p) = \frac{1}{N}\sum_{m=0}^{N-1-p} x_1(m)x_1(m+p)$$ Correlogram

Same as autocorrelation with extra 1/N term as summation done only on N terms

Correlogram



Correlogram

Consider the z transform:

$$S_N(z) = \frac{1}{N} X(z)X(z^{-1})$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)z^{-n} \sum_{m=0}^{N-1} x(m)z^{m}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} x(n)x(m)z^{m-n}$$

It can be shown that

$$S_N(z) = \sum_{p=-(N-1)}^{N-1} r_N(p)z^{-p}$$

$$= \sum_{p=0}^{N-1} r_N(p)[z^{-p} + z^{p}] - r_N(0)$$

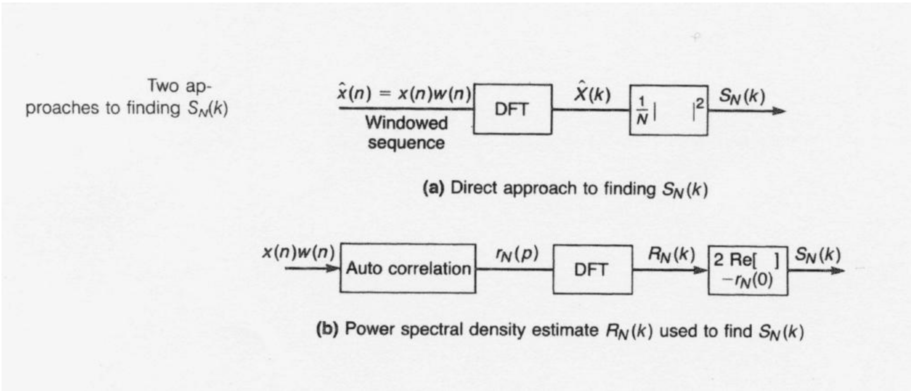
Therefore (see Notes):

$$S_N(k) = R_N(k) + R_N(k)^* - r_N(0)$$

and

$$S_N(k) = 2R_e\left[R_N(k)\right] - r_N(0)$$

Periodogram versus Correlogram

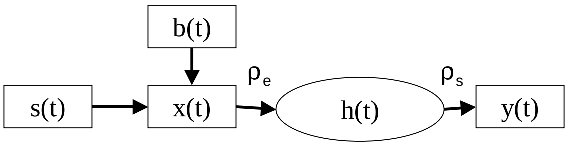


A parte on power density

Why use power spectral density?

$$\rho = SNR = \frac{P_{signal}}{P_{bruit}} = \frac{\int S(\nu)d\nu}{\int B(\nu)d\nu} = \frac{\sum_{k=0}^{N-1} S_N(k)}{\sum_{k=0}^{N-1} B_N(k)}$$

In a linear time invariant system (filter)



A parte on power density

Matched filtering: *optimizes SNR out* ρ_s

$$h(\nu) = e^{-j2\pi\nu T} \frac{s^*(\nu)}{B(\nu)}$$

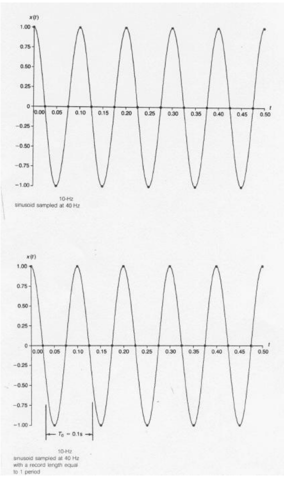
White noise case:

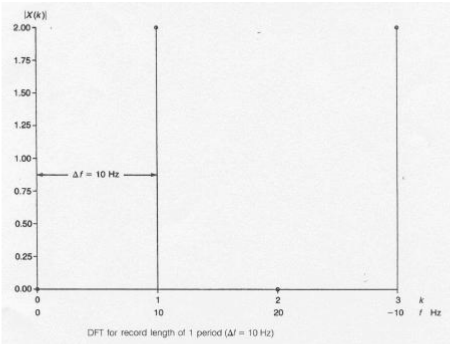
$$h(\nu) = e^{-j2\pi\nu T} s^*(\nu)$$

$$h(t) = s(T - t)$$ correlation

Windows for spectrum analysis

<u>Experiment:</u>



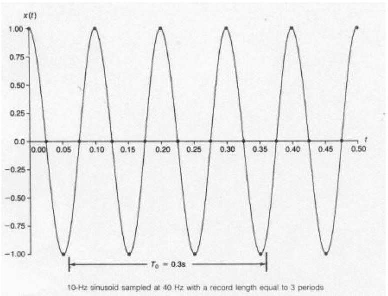


DFT Record Length of 1 period (Δf = 10 Hz)

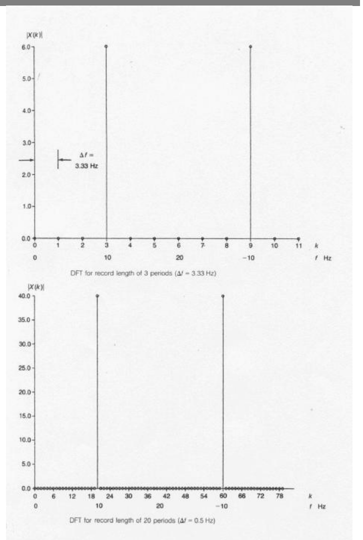
10 Hz signal sampled at 40 Hz analysed using a 0.1 s window

Windows for spectrum analysis

Experiment:



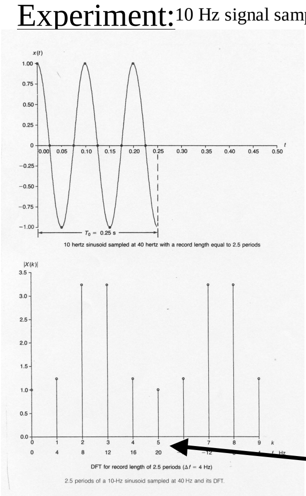
10 Hz signal sampled at 40 Hz analysed using a 0.3 / 2 s window

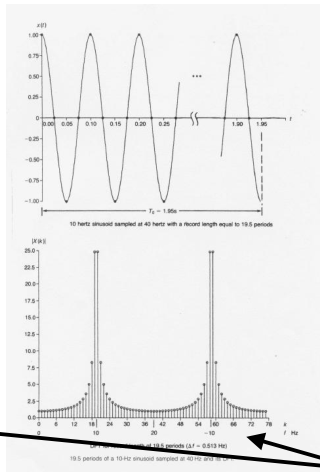


DFT Record Length of 1 period (Δf = 3.33 Hz / Δf = 0.5Hz)

Windows for spectrum analysis

Experiment: 10 Hz signal sampled at 40 Hz analysed using a 0.25 / 1.95s window





Check frequency content?

Check Levels

What is happening?

Leakage
or
Gibbs phenomenon

DFT Record Length of 1 period (Δf = 4 Hz / Δf = 0.513Hz)

Windows for spectrum analysis

Solution?

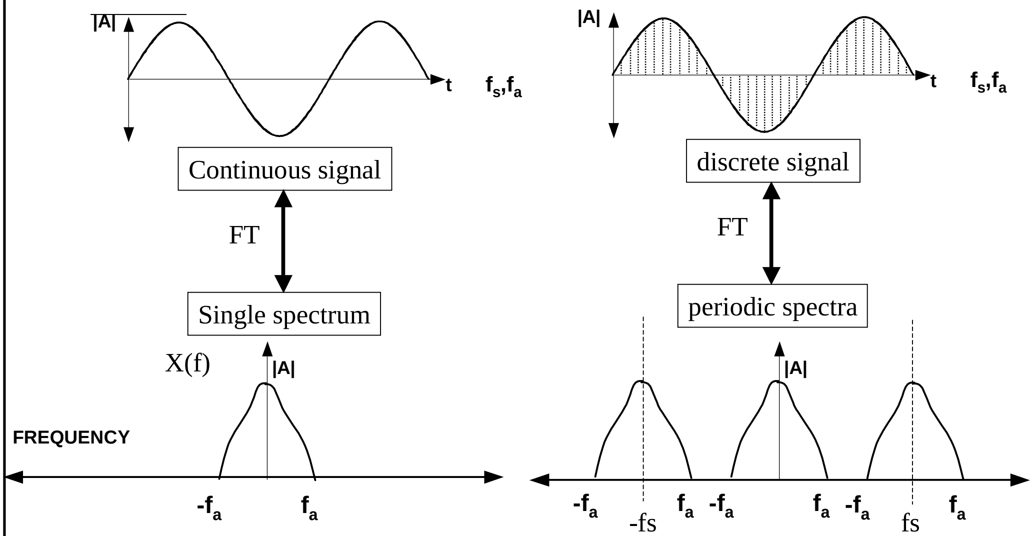
First clue: think about what a Discrete Fourier domain means in the time domain…

Second clue:

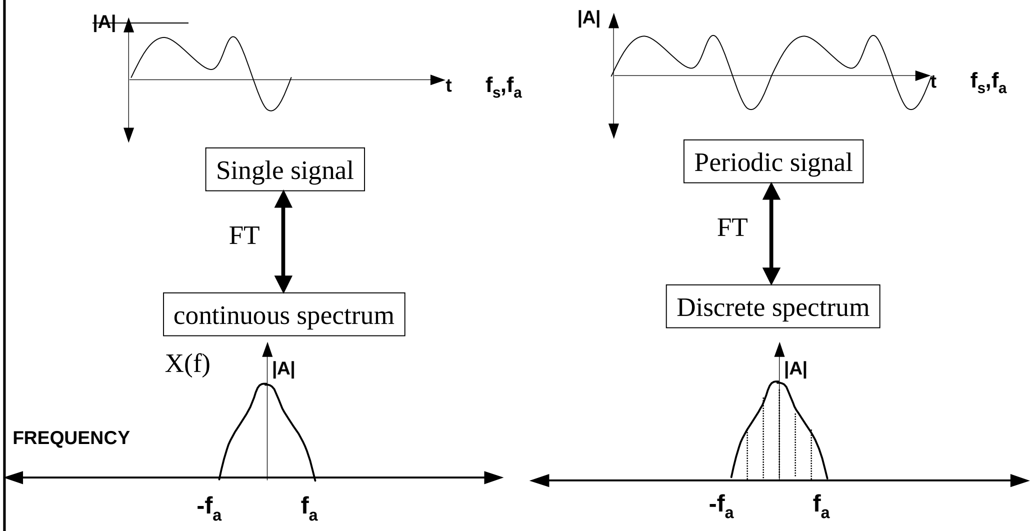
If you consider a integer number of periods what happens?

__What if it is a rational number of periods?

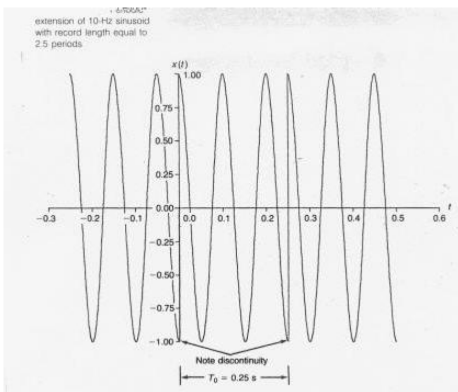
Reminder: Discreteness and periodicity

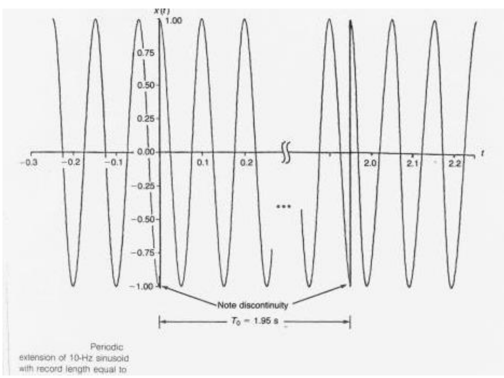


Reminder: Discreteness and periodicity



Windows for spectrum analysis





Windows for spectrum analysis

Case of real signals:

non-periodic

leakage is bound to happen unless the signal is the same on both side of the window of analysis:

Will not happen naturally for all signals and all windows

Will restrict the choice of windows / frequency resolution

Is unpractical for averaged periodogram estimation

Spectrum analysis 4.25

Windows for spectrum analysis

Solution is windowing the signals so that each end of the window have the same value!

Effect of windowing?

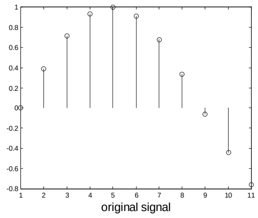
Theory:

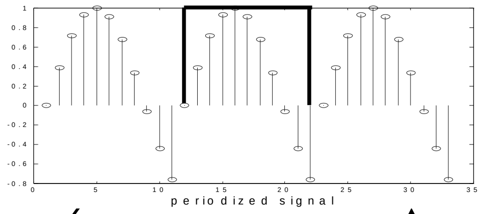
$$x_s = x(t)\sum_{k=0}^{\infty}\delta(t-kT) \qquad x_w = wx_s$$

$$X_s = \frac{1}{T_s}\sum_{k=0}^{\infty}X(f-kf_s) \qquad X_w = \underbrace{W * X_s}_{\text{convolution}}$$

Spectrum analysis 4.26

Square window



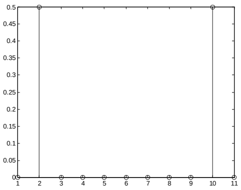


FT

$$X_w = W * X_s$$

convolution

FT



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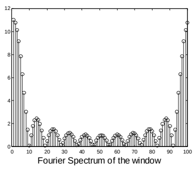

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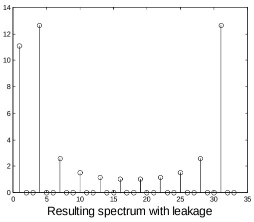
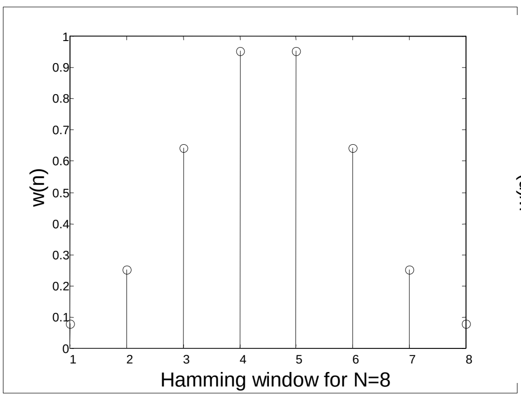
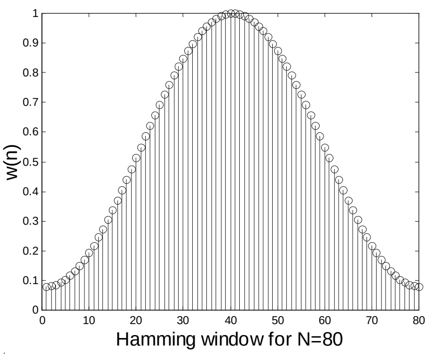


Optimised classical windows

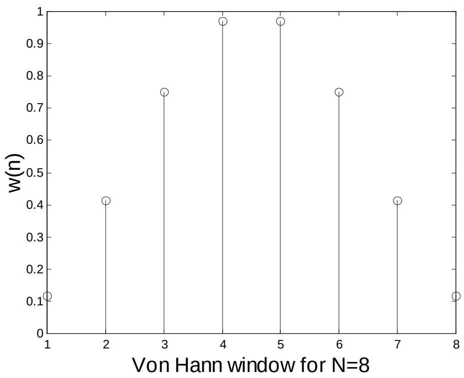
<u>Hamming:</u> $w(n) = 0.54 - 0.46 * \cos\left(\frac{2\pi n}{N-1}\right)$

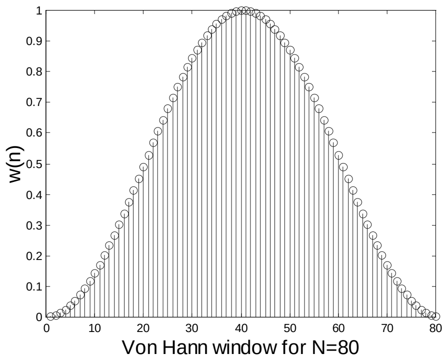




Optimised classical windows

<u>Hanning:</u> $w(n) = 0.50 - 0.50 * \cos\left(\frac{2\pi n}{N-1}\right)$



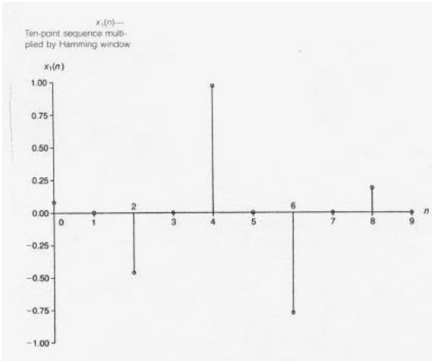


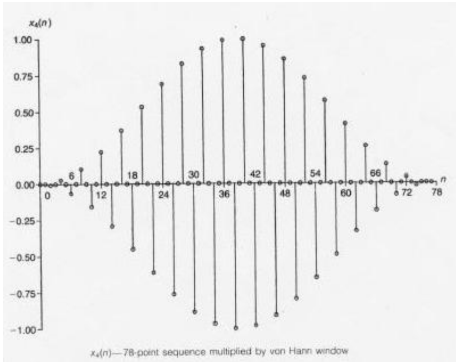
Optimised classical windows

<u>Difference between windows ?</u>

Window type	Mathematical expression	Sidelobes (dB)	Transition width	Stop band attenuation
Rectangular	$w(n) = 1, \quad 0 \le n \le N-1$	-13	0.9/N	-21
Hamming	$w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), \quad 0 \le n \le N-1$	-31	3.1/N	-44
Hanning	$w(n) = 0.50 - 0.40\cos\left(\frac{2\pi n}{N-1}\right), \quad 0 \le n \le N-1$	-41	3.3/N	-53
Blackman	$w(n) = 0.42 - 0.50\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right), \quad 0 \le n \le N-1$	-57	5.5/N	-74

Example

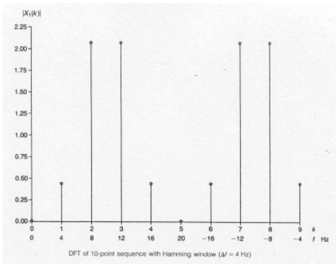




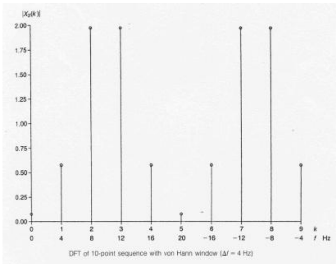
10 points sequence multiplied by hamming window

75 points sequence multiplied by hanning window

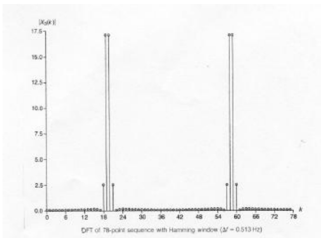
Example



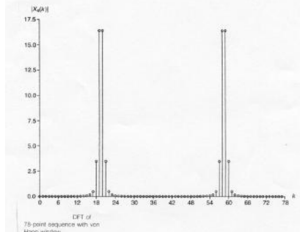
10 points DFT for Hamming window (Δf = 4 Hz)



10 points DFT for Hanning window (Δf = 4 Hz)



75 points DFT for Hamming window (Δf = 0.513 Hz)



75 points DFT for Hanning window (Δf = 0.513 Hz)