Discrete Fourier Transform

Dr Yvan Petillot

Section Contents

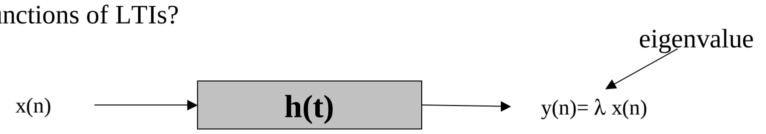
- LTIs and DFT
- Discrete Fourier Transform
- Relation between DFT and Fourier Transform
- Effects of various parameters on DFT
- Summary

Fourier Transform

- Used for spectral analysis
- Core to filtering (Convolution theorem)
- Core to signal modeling
- The Basic tool of digital signal processing

LTIs

Eigenfunctions of LTIs?



Are

$$x(n) = e^{jnw}, \quad -\infty < n < \infty$$

As:

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} h(k)e^{jw(n-k)}$$

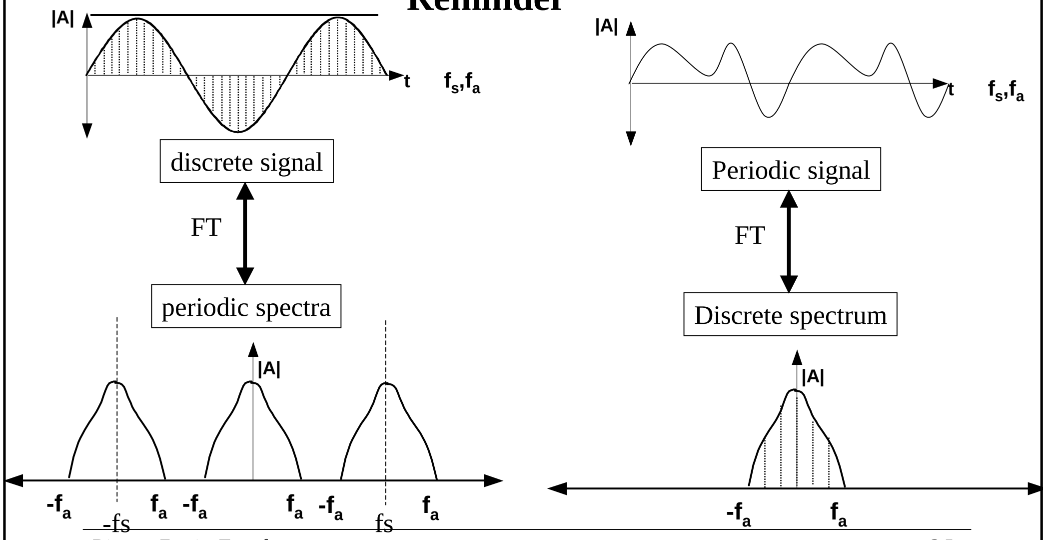
$$= e^{jwn} \sum_{k=-\infty}^{\infty} h(k)e^{-jwk} = e^{jwn}H(e^{jw}) = \lambda e^{jwn}$$

$$H(e^{jw}) = \sum_{k=-\infty}^{\infty} h(k)e^{-jwk}$$

Discrete Time Fourier Transform

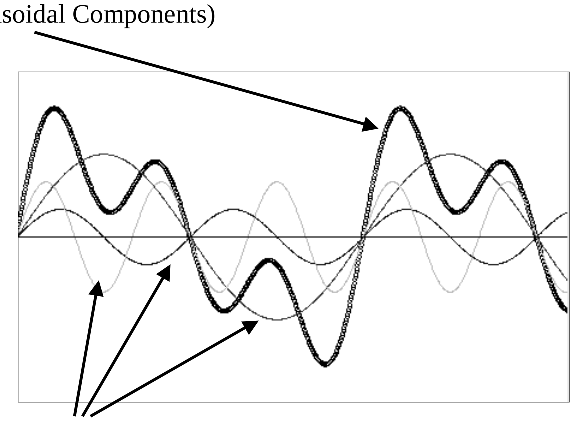
Discrete Fourier Transform

Reminder



Fourier Series Construction

Original Signal
(Sum of Sinusoidal Components)



Individual Pure Sinusoidal
Components

Discrete Fourier Transform

Discrete Fourier series

Give representation for periodic signals

$$x(t) = \sum_{k=-\infty}^{\infty} X(n)e^{jnw_0 t} \quad \text{where} \quad X(n) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{(-jnw_0 t)}dt$$

$w_0 = 2\pi / T$ where T is the period of the periodic signal

Real signal:

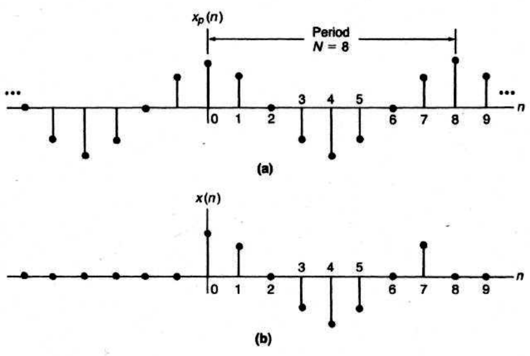
Non periodic, *finite length.* We cannot use Fourier Series!

a non-periodic finite sequence can be considered as *one period* of a periodic infinite sequence!

Discrete Fourier Transform

Example $\quad x_p(n + kN) = x(n), \quad 0 \le n \le N-1, \ -\infty \le k \le \infty$

A nonperiodic sequence $x(n)$ and a periodic sequence $x_p(n)$



$$x(n) = \begin{cases} x_p(n), & 0 \le n \le N-1 \\ 0 & \text{elsewhere} \end{cases}$$

Discrete Fourier Transform

Discrete Fourier Transform:

Discrete Fourier Series of the periodised signal $x_p(n)$

Use only one period of the resulting sequence for x(n)

Definition of the DFT

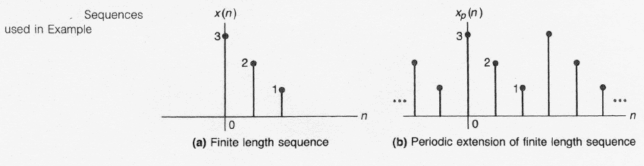
$$X(k) = \sum_{n=0}^{N-1} x_p(n)\ e^{-j2\pi \frac{kn}{N}}, \quad 0 \le k \le N-1$$

$$X(k) = \sum_{n=0}^{N-1} x(n)\ e^{-j2\pi \frac{kn}{N}}, \quad 0 \le k \le N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\ e^{j2\pi \frac{kn}{N}}, \quad 0 \le n \le N-1$$ Inverse DFT

Discrete Fourier Transform

Example



Calculate the DFT

Discrete Fourier Transform / Fourier Transform

Discrete Time Fourier Transform

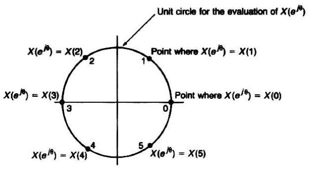
$$X(e^{j\theta}) = \sum_{n=0}^{N-1} x(n)\ e^{-jn\theta}$$

Discrete Fourier Transform

$$X(k) = \sum_{n=0}^{N-1} x(n)\ e^{-j(\frac{2\pi k}{N})n}$$

$$\theta \longrightarrow \frac{2\pi k}{N}$$

Relationship between the Fourier Transform and the DFT of a finite sequence for $N = 6$



Record length/Frequency resolution/Sampling frequency

$$x_s(k) = x(t)\sum_{k=-\infty}^{\infty}\delta(t - kT_s) = \sum_{k=0}^{N-1} x(kT_s)$$

Ts is the sampling frequency

$$X(k\Delta f) = \sum_{n=0}^{N-1} x(nT)\ e^{-j\frac{2\pi}{N}nk}$$

$$x(nT) = \frac{1}{N}\sum_{k=0}^{N-1} X(k\Delta f)\ e^{j\frac{2\pi}{N}nk}$$

Δf is the frequency spacing (resolution) of the DFT coefficients

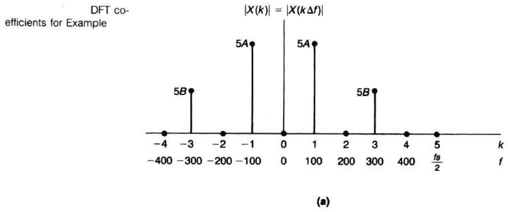
Record length/Frequency resolution/Sampling frequency

Example: $x(t) = A\cos(200\pi t) + B\cos(600\pi t)$

Sampled at 1kHz.

Find the period of the resulting sequence x(nT), i.e. N

Find the frequency resolution

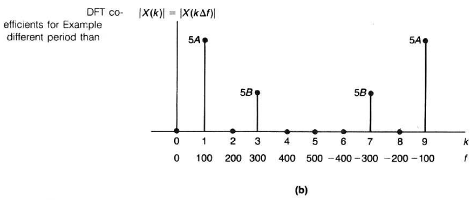


Record length/Frequency resolution/Sampling frequency

Frequency resolution: $\Delta f = \frac{f_s}{N} = \frac{1}{T_0}$

T_0 : signal duration

If x(nT) is made of frequency components spaced by less than Δf Hertz, the DFT will NOT represent them



Same example but k = 0,1,…,9

Notice folding around k = 5 = fs/2

Record length/Frequency resolution/Sampling frequency

An analogue signal of 200ms length is sampled at 2.5 kHz.

What is the maximum frequency present is aliasing is to be avoided?

What is the frequency resolution of the DFT?

What analogue frequencies are represented by the DFT?

Record length/Frequency resolution/Sampling frequency

Solution: An analogue signal of 200ms length is sampled at 2.5 kHz.

What is the maximum frequency present is aliasing is to be avoided?

Nyquist: max frequency = fs/2 = 1.25kHz

What is the frequency resolution of the DFT?

$$\Delta f = \frac{f_s}{N} = \frac{1}{T_0} = \frac{1}{0.2} = 5\text{Hz}$$

What analogue frequencies are represented by the DFT?

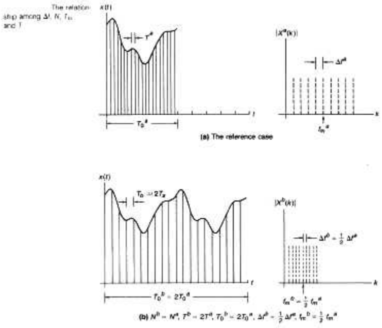
0, 5, 10, 15 ,…, 1250, -1245, …, -5 Hz

Examples of applications

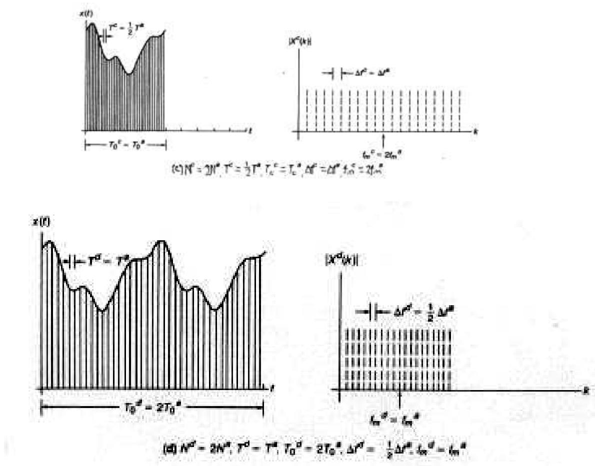
Hold N constant with two sampling periods T_a, T_b=2 T_a.

Effect?

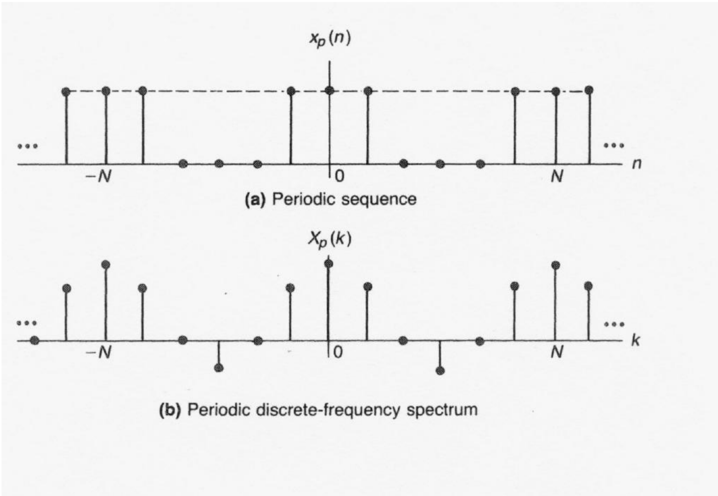
$$\Delta f_a = \frac{1}{T_0} = \frac{1}{NT_a}, \quad \Delta f_b = \frac{1}{T_0} = \frac{1}{NT_b} = \frac{1}{2NT_a} = \frac{\Delta f_a}{2}$$



Examples of applications



DFT analysis



DFT analysis example

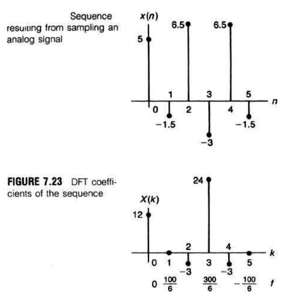
Ts= 0.01s

N	0	1	2	3	4	5
x(n)	5	-1.5	6.5	-3	6.5	-1.5

Frequency content of the sequence?

$$X(k) = \sum_{n=0}^{5} x(n)\ e^{-j(\frac{2\pi}{6})nk}$$

Frequency resolution?



Properties of the DFT

Linearity:

$$DFT\left[ax_1(n)+bx_2(n)\right]=\sum_{n=0}^{N-1}[ax_1(n)+bx_2(n)]\; e^{-j(\frac{2\pi}{N})nk}=aX_1(k)+bX_2(k)$$

Symmetry

if x(n) is a sequence of real numbers:

$$\mathrm{Re}\left[X(k)\right]=\mathrm{Re}\left[X(N-k)\right],\quad k=1,2,...,\frac{N}{2}-1,\quad N\text{ even}$$

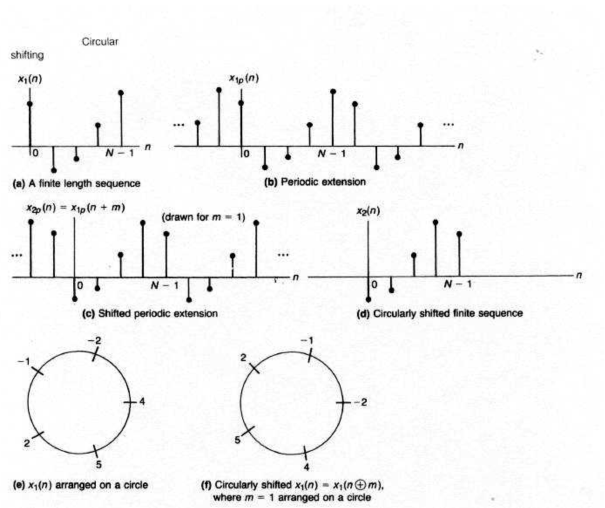
$$\mathrm{Im}\left[X(k)\right]=-\mathrm{Im}\left[X(N-k)\right],\quad k=1,2,...,\frac{N-1}{2},\quad N\text{ odd}$$

$$\left|X(k)\right|=\left|X(N-k)\right|$$

$$\angle X(k)=-\angle X(N-k)$$

Properties of the DFT

Circular shift



Properties of the DFT

Circular shift

$$x_2(n) = x_1(n+m)$$

$$X_2(k) = X_1(k)e^{j\frac{2\pi}{N}km}$$

Demonstration

$$x_2(n) = x_1(n+m) = x_1(n) * \delta(n+m)$$

$$X_2(k) = X_1(k)e^{j\frac{2\pi}{N}km}$$

Direct demonstration left as an exercise

Properties of the DFT

Alternate IDFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\ e^{j2\pi\frac{kn}{N}}, \qquad 0 \le n \le N-1$$

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k)\ e^{-j2\pi\frac{kn}{N}}\right]^*$$

Duality

$$x(n) \leftrightarrow X(k)$$

$$\frac{1}{N}X(N) \leftrightarrow x(-k) \qquad \text{Why } \frac{1}{N}\text{?}$$

PARSEVAL THEOREM

$$\sum_{n=-\infty}^{\infty}|x(n)|^2 = \frac{1}{N}\sum_{k=-\infty}^{\infty}|X(k)|^2$$

Properties of the DFT

Example: $x(n) = \cos(2\pi n/N), \ n = 0,1,\ldots, N-1$

$x(n) = 0$ elsewhere

Complete the relationship using the duality principle:

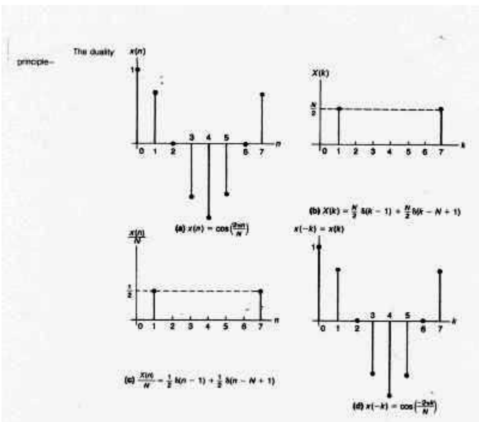
$x(n) = \cos(2\pi n/N) \leftrightarrow \quad X(k) = ?$

$\frac{1}{N} X(n) = ? \leftrightarrow \quad x(-k) = ?$

Properties of the DFT

Example: $x(n) = \cos(2\pi n/N), \ n = 0,1,\ldots, N-1$

$x(n) = 0$ elsewhere



Discrete convolution

- Remember: Filtering is a linear convolution!

$$c(n) = x_1(n) * x_2(n) = \sum_{m=-\infty}^{\infty} x_1(m) x_2(n-m)$$ Discrete Linear convolution

- Infinite signals uncommon!

- Can we do the same with finite, periodic signals?

Discrete convolution

Linear Convolution

$$c(n) = x_1(n) * x_2(n) = \sum_{m=-\infty}^{\infty} x_1(m) x_2(n-m)$$ Linear convolution

Periodic Convolution

$$X_{1p}(k) = \frac{1}{N} \sum_{k=0}^{N-1} x_{1p}(n)\; e^{-j2\pi\frac{kn}{N}}$$

$$X_{2p}(k) = \frac{1}{N} \sum_{k=0}^{N-1} x_{2p}(n)\; e^{-j2\pi\frac{kn}{N}}$$

$$\boxed{X_{3p}(k) = X_{1p}(k) X_{2p}(k)}$$

$$X_{3p}(k) = X_{1p}(k) X_{2p}(k) = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x_{1p}(n)\, x_{2p}(m)\; e^{-j(\frac{2\pi k}{N})(n+m)}$$

$$X_{3p}(k) = \sum_{m=0}^{N-1} x_{3p}(n)\; e^{-j(\frac{2\pi k}{N})(n)}$$

$$\boxed{x_{3p}(n) = \sum_{m=0}^{N-1} x_{1p}(m) x_{2p}(n-m)}$$

Periodic convolution

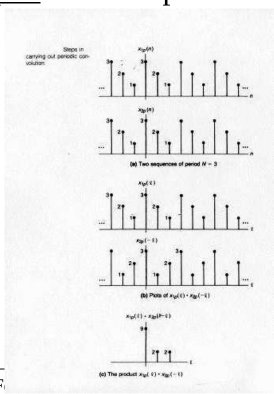
Periodic Convolution: Same as linear convolution but over one period

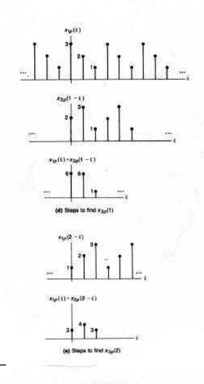
Periodic convolution

Periodic Convolution

$$x_{3p}(n) = \sum_{m=0}^{N-1} x_{1p}(m)x_{2p}(n-m) \quad \overset{\text{DFT}}{\longleftrightarrow} \quad X_{3p}(k) = X_{1p}(k)X_{2p}(k)$$

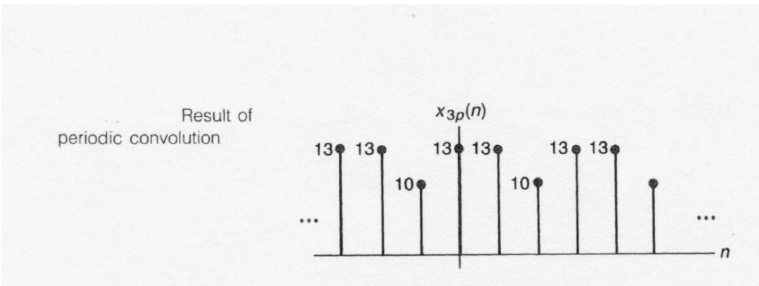
Example: Find the periodic convolution of sequence below





Periodic convolution

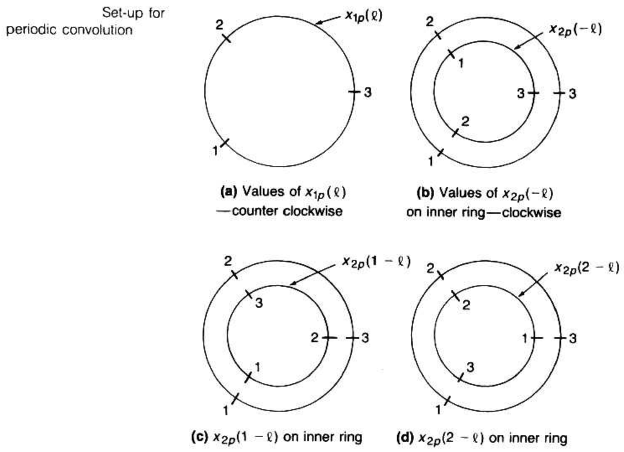
Final result:



A bit tedious?

Periodic convolution

<u>Other solution:</u>



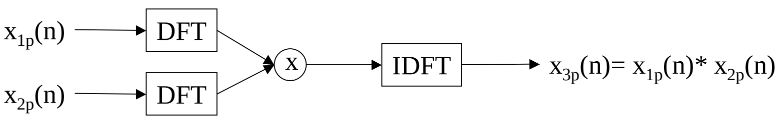
Periodic convolution

<u>Summary:</u>

Periodic convolution of two sequences can be obtained by:

- Remaining in the time domain and using the convolution sum directly.

- Moving into the frequency domain using the following scheme:



Circular convolution

New Problem:

Can we perform linear convolution with finite length signals using DFTs ?

If we use DFT, we are dealing with sampled spectrum which implies … periodic signals!

Why not use the sum definition and forget about DFTs?

Circular convolution

A Parte:

Number of points	Direct DFT		Radix 2 FFT		Direct Sum and adds convolution	
	Complex multiplies	Complex additions	Complex multiplies	Complex additions	Complex multiplies	Complex additions
N	N^2	N^2-N	(N/2) log2(N)	Nlog2(N)	2 N^2	N^2
4	16	12	4	8	32	16
16	256	240	32	64	512	256
64	4096	4032	192	384	8192	4096
256	65536	65280	1024	2048	131072	65536
1024	1048576	1047552	5120	10240	2097152	1048576

DFTs in FFTs form are good for you!

Circular convolution

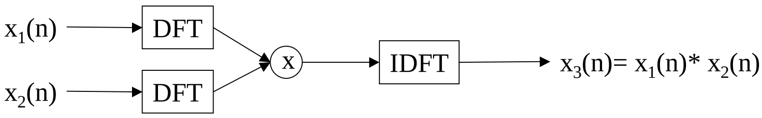
Definition:

Consider $x_1(n)$ and $x_2(n)$ and $x_{1p}(n)$ and $x_{2p}(n)$ their periodic equivalent.

$$x_{3p}(n) = \sum_{m=0}^{N-1} x_{1p}(m) x_{2p}(n-m)$$

$$\boxed{x_3(n) = \left[\sum_{m=0}^{N-1} x_{1p}(m) x_{2p}(n-m)\right]_{\text{one period}} = x_1(n) * x_2(n)}$$ Circular convolution

$$X_{3p}(k) = X_{1p}(k) X_{2p}(k)$$



Circular convolution

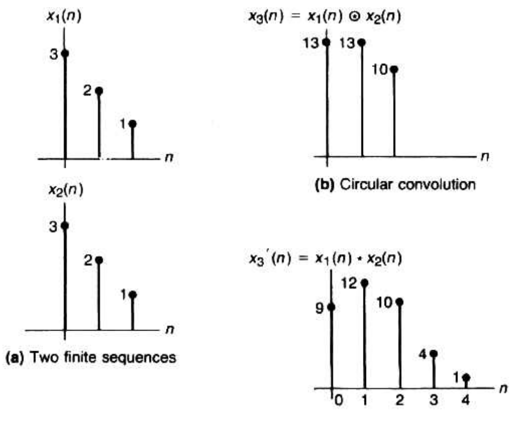
Is it what we want?

Different values

Different length

Linear convolution:
N1+N2-1 length

Circular convolution:
max(N1,N2).



(a) Two finite sequences

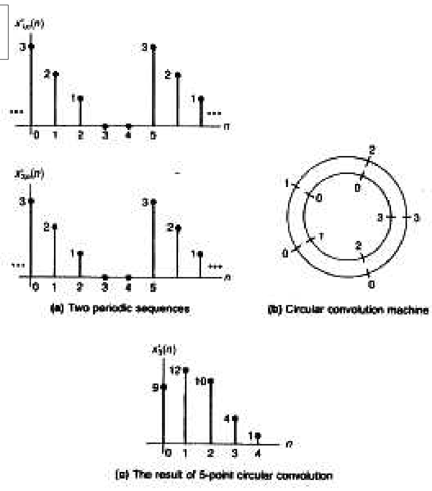
(b) Circular convolution

(c) Linear convolution

Circular convolution

Solution?

Forcing linear and Circular Convolution to be equivalent



Frequency Convolution

<u>Use duality properties</u>

$$x_1(n) * x_2(n) \leftrightarrow X_1(k)X_2(k)$$

$$x_1(n)x_2(n) \leftrightarrow \frac{1}{N} X_1(k) * X_2(k)$$

Correlation

- Measure of similarity
- Target detection & classification
- Noise rejection
- Signal Modeling
- Related to the Power Spectral Density (PSD)
- Related to statistical description of signals (noise)

Correlation

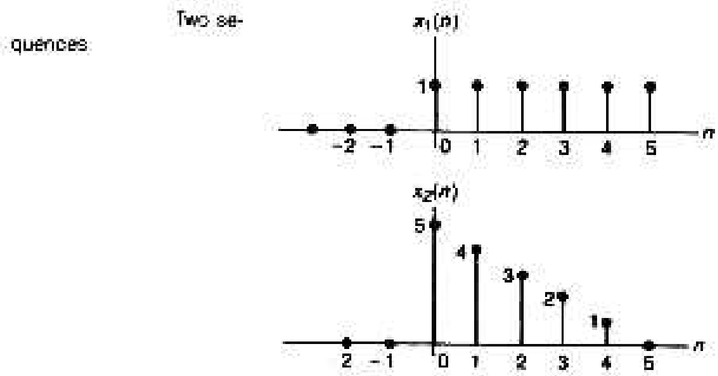
Same as convolution but with no time reversal.

$$R_{x_1x_2}(n) = x_1(n) \otimes x_2(n) = \sum_{m=-\infty}^{\infty} x_1(m)x_2(n+m)$$ Crosscorrelation

$$R_{x_1x1}(n) = x_1(n) \otimes x_1(n) = \sum_{m=-\infty}^{\infty} x_1(m)x_1(n+m)$$ Autocorrelation

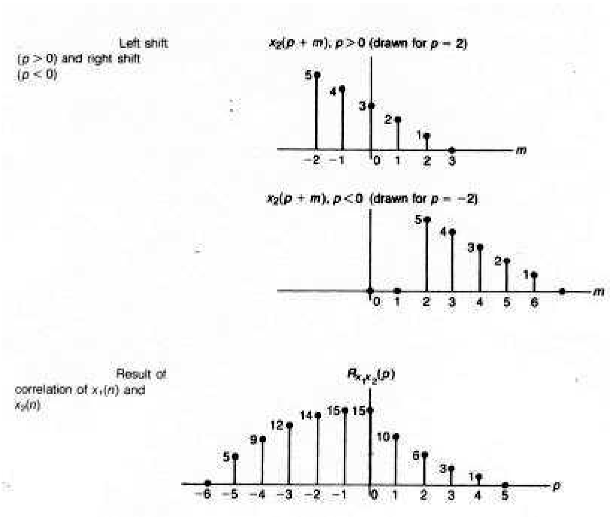
Correlation

Example



Correlation

Example



Correlation

<u>Properties</u>

$R_{x_1x_1}(n) = R_{x_1x_1}(\text{-}n)$ Autocorrelation is always even

$R_{x_1x_2}(n) = R_{x_2x_1}(\text{-}n)$ Cross correlation case

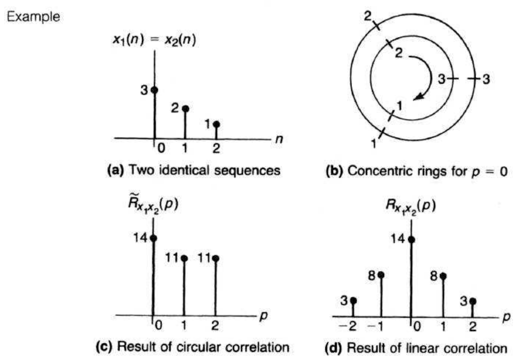
$$R_{x_1x_2}(p) = x_1(n) * x_2(-n)$$
$$R_{x_2x_1}(p) = x_1(-n) * x_2(n)$$

Convolution

Correlation and DFTs

<u>Circular correlation:</u>

$$\tilde{R}_{x_1x_2}(n) = \left[\sum_{m=0}^{N-1} x_{1p}(m)x_{2p}(n+m)\right]_{\text{one period}} = \sum_{m=0}^{N-1} x_1(m)x_2(n+m)$$



(a) Two identical sequences (b) Concentric rings for $p = 0$

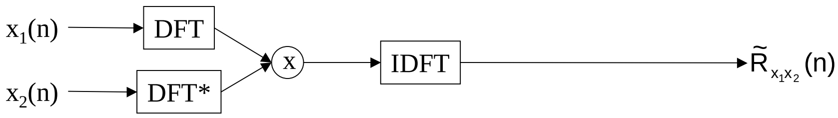
(c) Result of circular correlation (d) Result of linear correlation

Correlation and DFTs

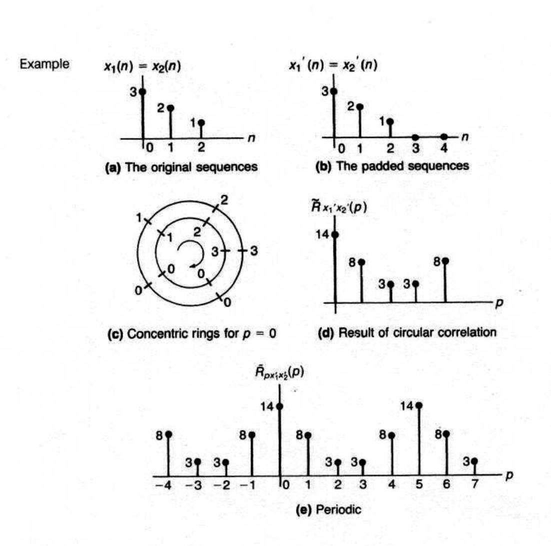
Can we perform circular correlation with DFTs ?

$$\tilde{R}_{x_1x_2}(n) = \left[\sum_{m=0}^{N-1} x_{1p}(m)\,x_{2p}(n+m)\right]_{\text{one period}} = x_1(n) \otimes x_2(n)$$ Circular correlation

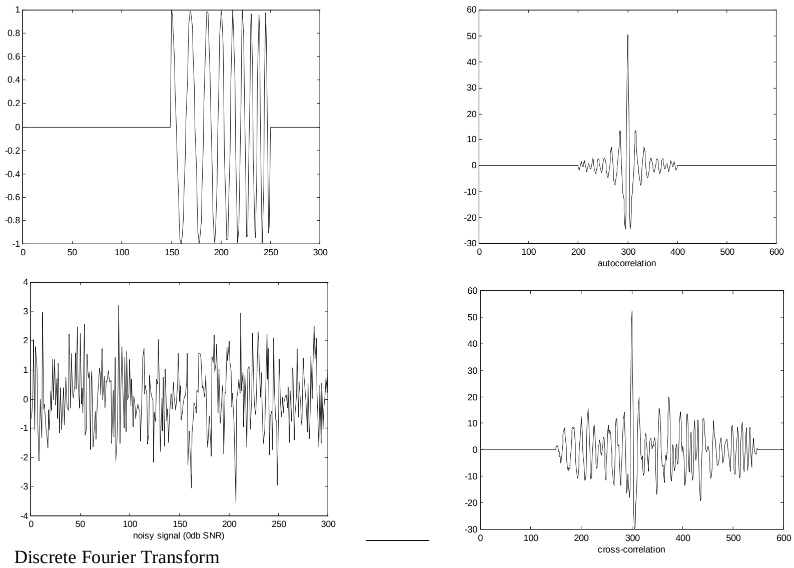
$$\text{DFT}\left[\tilde{R}_{x_1x_2}(n)\right] = X^{*}_{1p}(k)\,X_{2p}(k)$$



Linear Correlation and DFTs



Example of application



Learning outcomes

Discrete Fourier Transform, definition and properties

DFT analysis

Influence of parameters on frequency estimation

Discrete convolution: linear / circular

Discrete correlation: linear / circular