Circularly symmetric phase filters for control of primary third-order aberrations: coma and astigmatism

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A quartic phase retardation function is described that reduces the variation of the intensity of the focal point of incoherent imaging systems suffering from primary third-order aberrations limited to coma and astigmatism. Corresponding modulation transfer functions are shown to remain practically invariant for moderate amounts of coma and astigmatism.

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1. INTRODUCTION

Until recently, the control of defocus and reduction of third-order aberrations, such as spherical aberration, were carried out by means of amplitude filters; of special interest are the annular apodizers derived by Ojeda-Castañeda *et al.*[1,2] With the widespread introduction of digital signal processing techniques, phase masks offer the advantage of improved optical resolution and light gathering as is the case with wavefront-coded imaging systems.[3] In this paper we extend our previous work in which we derived a radially symmetric quartic phase mask that reduces the sensitivity of incoherent imaging systems to variations in defocus and spherical aberrations.[4,5] Using a similar process, based on the evaluation of the Strehl ratio, we show that quartic filters (QFs) also yield reduced sensitivity to coma and astigmatism. Practically, the QF has been successfully employed to increase the depth of focus of an optical system for tracking stars.[6]

The normalized axial intensity of the point-spread function at the image plane of an optical system is given by

$$I = C\left|\int_0^{2\pi}\int_0^1 P(r)\exp[\mathrm{i}kW(r,\theta)]r\mathrm{d}r\mathrm{d}\theta\right|^2, \qquad (1)$$

where the exit pupil plane is represented in polar coordinates (r,θ), k is the wavenumber $(2\pi/\lambda)$, C is a normalizing constant, $W(r,\theta)$ is the wave aberration function, and $P(r)$ denotes the pupil function with $P(r)=1$ for $0 \leqslant r \leqslant 1$, and $P(r)=0$ elsewhere. The radial coordinate r is normalized to the radius of the pupil. Rotationally symmetric phase filters,$\Phi(r)$, that are intended to reduce third-order aberrations are placed in the pupil plane; thus Eq. (1) becomes

$$I_i = C\left|\int_0^{2\pi}\int_0^1 \exp\{\mathrm{i}k[W(r,\theta)+\Phi(r)]\}r\mathrm{d}r\mathrm{d}\theta\right|^2. \qquad (2)$$

For an imaging system that suffers from primary third-order aberrations restricted to coma (W_{31}) and astigmatism (W_{22}), the function $W(r,\theta)$ is written in terms of Zernike polynomials as

$$W(r,\theta) = W_{31}r^3\cos\theta + W_{22}r^2\cos^2\theta. \qquad (3)$$

This last expression can be written in a convenient form by employing a change of variable in the radial coordinate r,

$$r^2 = \xi + 1/2. \qquad (4)$$

In the new coordinates (ξ,θ), the aberration function is rewritten as

$$\tilde{W}(\xi,\theta) = W_{31}(\xi+1/2)^{3/2}\cos\theta + W_{22}(\xi+1/2)\cos^2\theta. \qquad (5)$$

The analytical expression of the intensity in Eq. (2) becomes

$$I = C\left|\int_0^{2\pi}\int_{-1/2}^{1/2} \exp\{\mathrm{i}k[\tilde{W}(\xi,\theta)+\tilde{\Phi}(\xi,\theta)]\}\mathrm{d}\xi\mathrm{d}\theta\right|^2, \qquad (6)$$

where the tilde denotes functions expressed in the new radial coordinate ξ.

The present work is restricted to the study of imaging systems in which only one residual third-order aberration, such as coma or astigmatism, is present. Third-order spherical aberration was an object of study in Refs. 4 and 5, in which, by means of the same methodology, logarithmic and QFs were derived and exhibited high tolerance to spherical aberration.

2. COMA

Coma is introduced by misalignment or tilt of optical elements or as an inherent aberration associated with off-axis imaging. It can be problematic in Newtonian telescopes with nonhyperbolic mirrors,[7] even for small fields of view.

From Eq. (6), the central intensity at the image plane of an optical system suffering from coma is given by

$$I(W_{31}) = \left| \int_0^{2\pi} \int_{-1/2}^{1/2} \exp\{ik[\tilde{\Phi}(\xi) + W_{31}(\xi + 1/2)^{3/2} \cos\theta]\} \mathrm{d}\xi \mathrm{d}\theta \right|^2, \tag{7}$$

where the normalizing constant is neglected. By integrating Eq. (7) over θ, we obtain

$$I(W_{31}) = \left| \int_{-1/2}^{1/2} J_0(kW_{31}(\xi + 1/2)^{3/2}) \exp[ik\tilde{\Phi}(\xi)] \mathrm{d}\xi \right|^2, \tag{8}$$

where J_0 denotes a zero-order Bessel function of the first kind.

The aim of this work is to find an expression for the phase function $\tilde{\Phi}(\xi)$, which yields an irradiance distribution with reduced sensitivity to coma. To evaluate the integral in Eq. (8), the stationary phase method[8] is employed, which assumes that the major contribution to the integral occurs at the point ξ_s, called the stationary point, where the phase function of the integrand is varying slowly. The intensity in Eq. (8) can be approximated by

$$I(W_{31}) \approx A \frac{[J_0(kW_{31}(\xi_s + 1/2)^{3/2})]^2}{|\tilde{\Phi}''(\xi_s)|}, \tag{9}$$

where A is a real-valued multiplicative constant and the double prime represents the second derivative of the phase function. The stationary point ξ_s is given by

$$\tilde{\Phi}'(\xi)_{\xi_s} = 0. \tag{10}$$

Approximation (9) is valid only if the phase function $\tilde{\Phi}(\xi)$ satisfies the following condition[9]:

$$\frac{3}{2} kW_{31} \sqrt{\xi + \frac{1}{2}} \left| \frac{J_1\left(kW_{31}\left(\xi + \frac{1}{2}\right)^{\frac{3}{2}}\right)}{J_0\left(kW_{31}\left(\xi + \frac{1}{2}\right)^{\frac{3}{2}}\right)} \right| < \sqrt{\frac{k\tilde{\Phi}''(\xi_s)}{4\pi}} \quad \text{for} \quad \left| \xi_s \pm \frac{1}{2} \right| > \sqrt{\frac{4\pi}{k\tilde{\Phi}''(\xi_s)}}, \tag{11}$$

where J_1 denotes a first-order Bessel function of the first kind.

When the stationary point is outside the boundary of integration, $-1/2 < \xi_s < 1/2$, both the amplitude of the intensity at the focal point and the optical resolution are dramatically reduced. Note that when the stationary point is near the integration domain ($\xi_s \approx -1/2$), approximation (9) is no longer valid as the boundary effects become important and should be taken into account.

According to the expression of intensity in approximation (9), the effect of coma, W_{31}, is explicitly manifest only in the argument of the Bessel function as a factor of $(\xi_s + 1/2)^{3/2}$. It is therefore possible to reduce the effect of coma by choosing a particular phase function $\tilde{\Phi}(\xi)$ that yields a stationary point that has a value that is close to the boundary point $\xi_s \approx -1/2$ while satisfying the condition given by inequality (11). Practically, in the presence of moderate amounts of coma and when the stationary point is located about the integration boundary $\xi_s \approx -1/2$, the argument of the Bessel function in approximation (9) becomes sufficiently small so that it can be approximated by

$$I(W_{31}) \approx A \frac{[J_0(kW_{31}(\xi_s + 1/2)^{3/2})]^2}{|\tilde{\Phi}''(\xi_s)|} \approx A \frac{1}{|\tilde{\Phi}''(\xi_s)|}. \tag{12}$$

This expression for the intensity is comparable to Eq. (5) of Ref. 5, where the phase retardation function $\tilde{\Phi}(\xi)$ that yields enhanced axial intensity invariance is given by a QF [i.e., $\tilde{\Phi}''(\xi) = \text{const.}$], as also reported by Zalvidea *et al.*,[10] to reduce defocus aberration.

The QF is expressed in the modified radial coordinate ξ as

$$\tilde{\Phi}(\xi) = \alpha\xi^2 + \alpha_0\xi, \tag{13}$$

where the constants α and α_0 are real valued. In the radial coordinate r, the phase function is written as

$$\Phi(r) = \alpha(r^2 - 1/2)^2 + \alpha_0(r^2 - 1/2). \tag{14}$$

Note that the parameter α_0 controls the amount of focal shift along the optical axis. The stationary point ξ_s is then deduced by substituting Eq. (13) into Eq. (10) to yield

$$\xi_s = -\frac{\alpha_0}{2\alpha}, \tag{15}$$

which can be substituted into approximation (9) to give

$$I(W_{31}) \approx \frac{A}{2\alpha} \left[J_0\left(kW_{31}\left(-\frac{\alpha_0}{2\alpha} + 1/2 \right)^{3/2} \right) \right]^2. \tag{16}$$

The ratio of the QF parameters α/α_0 determines the variation of the axial intensity $I(W_{31})$ for increasing values of coma W_{31}.

Large values of α and α_0 produce broad point-spread functions with low axial intensity, which results in a reduction of the magnitude of the modulation transfer function (MTF) and its cutoff frequency. Therefore practical optical systems based on this technique require a trade-off between tolerance to aberrations and reduction in resolution. The values of the QF parameters are obtained using algorithms based on user-defined merit functions with constraints in the magnitude of the axial intensity and/or the magnitude of the MTF. In the present work, the numerical value of α and α_0 are obtained by setting the MTF constraints to an effective cutoff frequency of 0.5 while attaining a maximum average value in its magnitude. After optimization, a QF was designed with $\alpha=15$ and $\alpha_0=9.75$. These values result in a stationary point $\xi_s=-0.325$, which satisfies the boundary conditions of inequality (11), and allow a reduction in the effect of coma on the Strehl ratio as shown in Fig. 1. Further assessment of the performance achieved by the QF is carried out by numerical simulation of the MTF subject to various amounts of coma, W_{31}. To appreciate the performance of the QF, the MTFs of an optical system with and without a QF are shown in Fig. 2. Since the designed QF yields a MTF with a reduction in the effective cutoff frequency, another set of comparison figures are included with a stopped-down clear aperture that has the same effective maximum spatial-frequency bandwidth as the QF. It can be observed that both the QF and stopped-down aperture MTFs remain practically constant for $0<W_{31}<3$ waves and offer improved tolerance to coma aberration at a cost of reduced resolution. To complement the analysis, point-spread functions were calculated for an optical system with and without the QF (see Fig. 3).

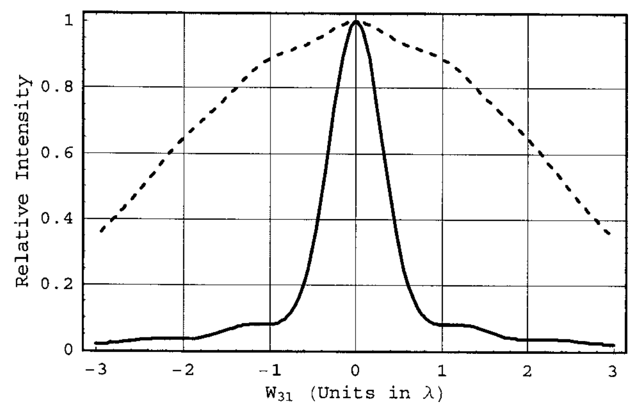

Fig. 1. Variation of the central intensity as a function of coma W_{31} with a QF with $\alpha=15$ and $\alpha_0=9.75$ (dashed curve) and without a QF (solid curve). The normalization coefficient for the QF intensity is 0.045.

Although the benefit of employing the QF over a stopped-down aperture is not apparent for coma aberration, it will be demonstrated in Section 3 that the use of this QF is advantageous in the presence of astigmatism. It is also worth noting that the reduction in the effective cutoff frequency will reduce the effects of aliasing in many pixelated imaging systems.

3. ASTIGMATISM

In a simple optical system, astigmatism increases with the square of the field angle. It is a common aberration found in human eyes, due to anamorphism of the cornea, and is normally accompanied by defocus. Special ophthalmic lenses or refractive surgery are used to compensate for its effects. We propose an alternative method, based on a phase filter, that produces point-spread functions that are practically unaffected by astigmatism.

The variation of the intensity at the focal point of an imaging system suffering only astigmatism, W_{22}, is given by

$$I(W_{22}) = \left| \int_0^{2\pi} \int_{-1/2}^{1/2} \exp\{ik[\tilde{\Phi}(\xi) + W_{22}(\xi+1/2)\cos^2\theta]\} d\xi d\theta \right|^2. \tag{17}$$

Using trigonometric identities and integrating over θ, the intensity becomes

$$I(W_{22}) = \left| \int_{-1/2}^{1/2} J_0(kW_{22}(\xi+1/2)/2) \times \exp\{ik[\tilde{\Phi}(\xi) + (W_{22}/2)\xi]\} d\xi \right|^2, \tag{18}$$

which can in turn be approximated by means of the stationary phase method used in Section 2, as

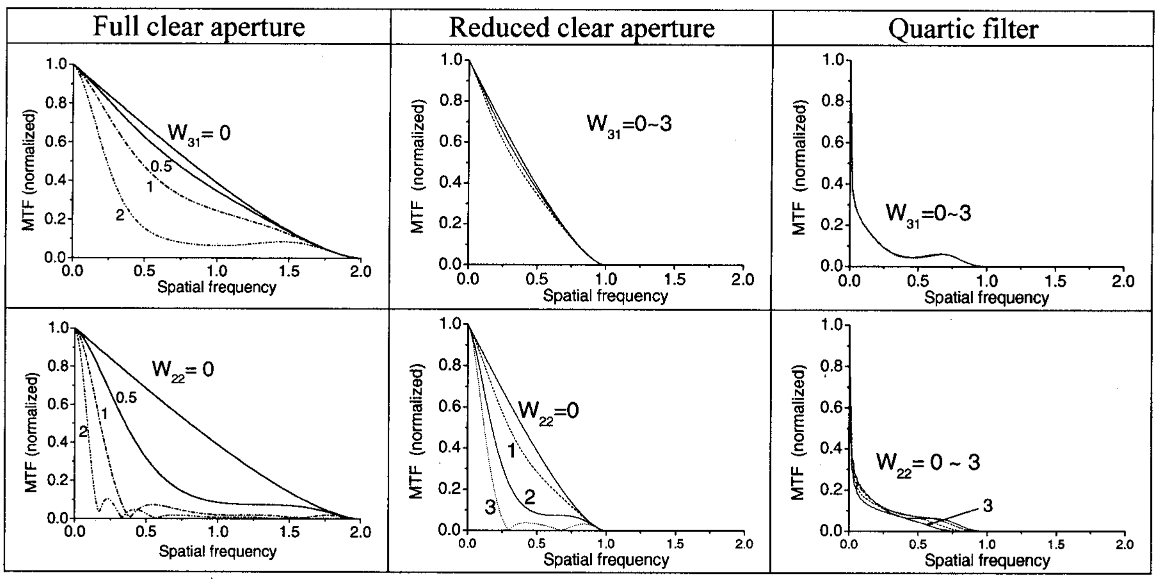

Fig. 2. Computed MTFs for imaging systems suffering from coma W_{31} and astigmatism W_{22} for clear aperture (left), reduced aperture (center), and a full aperture with a QF with $\alpha=15$ and $\alpha_0=9.75$ (right). Aberration W is in units of wavelength.

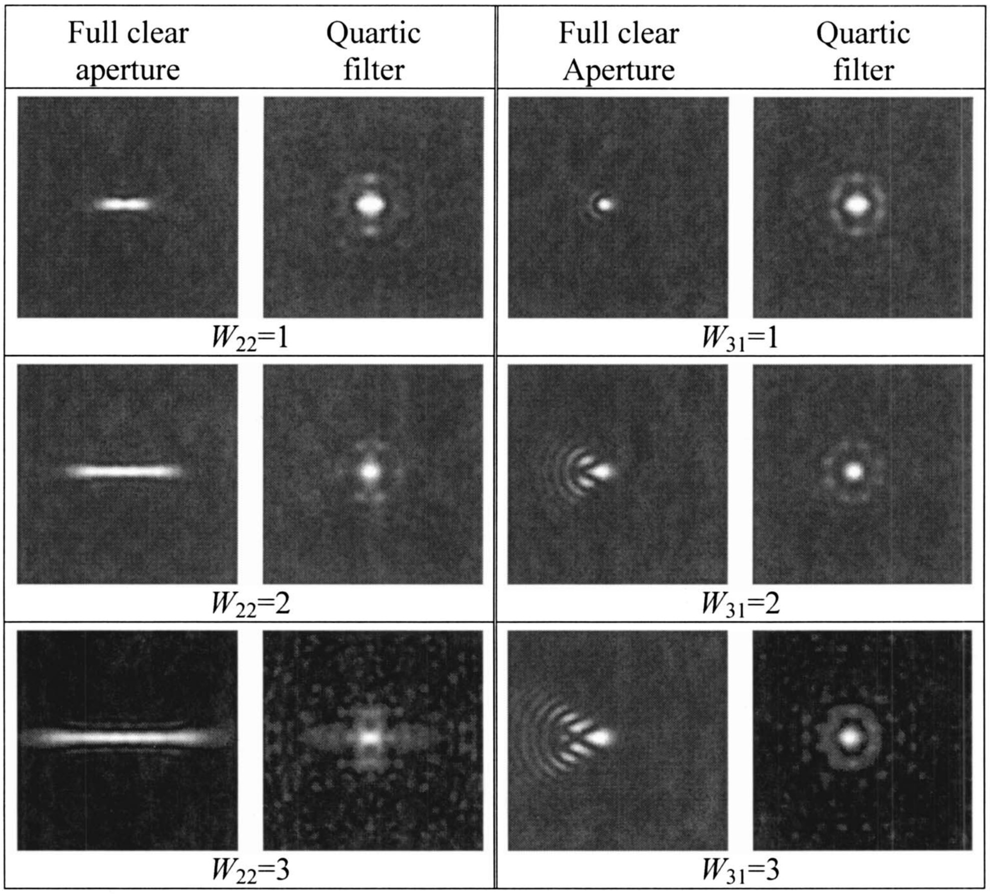

Fig. 3. Computed point-spread functions with and without QF ($\alpha=15$ and $\alpha_0=9.75$) in the presence of coma W_{31} (right) and astigmatism W_{22} (left). Aberration W is in units of wavelength.

$$I(W_{22}) \approx B \frac{\left[J_0\left(\frac{1}{2} k W_{22} \left(\xi_s + \frac{1}{2} \right) \right) \right]^2}{|\tilde{\Phi}''(\xi_s)|}, \tag{19}$$

where B is a real-valued constant and ξ_s is the stationary point defined by

$$\tilde{\Phi}'(\xi)_{\xi_s} + W_{22}/2 = 0. \tag{20}$$

Approximation (19) is valid only if the phase function $\tilde{\Phi}$ satisfies the following boundary condition:

$$\frac{1}{2} k W_{22} \left| \frac{J_1\left(\frac{1}{2} k W_{22} \left(\xi + \frac{1}{2} \right) \right)}{J_0\left(\frac{1}{2} k W_{22} \left(\xi_s + \frac{1}{2} \right) \right)} \right| < \sqrt{\frac{k \tilde{\Phi}''}{4\pi}}$$

$$\text{for } \left| \xi_s \pm \frac{1}{2} \right| > \sqrt{\frac{4\pi}{k \tilde{\Phi}''}}. \tag{21}$$

According to approximation (19), the expression for the intensity contains astigmatism, W_{22}, only in the argument of the Bessel function that has a coefficient of ($\xi_s + 1/2$). Note that Eq. (20) is distinct from Eq. (10) only by the additional term $W_{22}/2$. As in the previous analysis for coma, the same phase retardation $\tilde{\Phi}(\xi)$ that describes quartic phase filters could also attenuate the effect of astigmatism at the focal point when the stationary point satisfies the boundary condition given by inequality (21). The expression of the intensity in approximation (19) is rewritten as

$$I(W_{22}) \approx \frac{B}{2\alpha} \left[J_0\left(\frac{1}{2} k W_{22} \left(-\frac{W_{22}}{4\alpha} - \frac{\alpha_0}{2\alpha} + 1/2 \right) \right) \right]^2. \tag{22}$$

For sufficiently large values of α and modest astigmatism, the term $W_{22}/4\alpha$ can be neglected and the expression of the intensity as a function of W_{22} becomes analogous to that for coma W_{31} [see approximation (16)]. It should be noted, however, that high values of α that compensate for large amounts of astigmatism also result in a suppressed MTF and lower contrast in the image so that careful assessment in the selection of the parameter of the QF is necessary to trade contrast against aberration invariance.

The relative central intensity produced by the QF with the same parameters as previously, $\alpha=15$ and $\alpha_0=9.75$, is shown in Fig. 4 for various amounts of astigmatism. The dependence of the position of the stationary point on astigmatism is clearly manifested by the asymmetry of the relative intensity at either side of $W_{22}=0$. For decreasing negative W_{22}, the stationary point is moved toward the boundary at $\xi=-1/2$, requiring a higher value than $\alpha=15$ to compensate for both the boundary effects and the

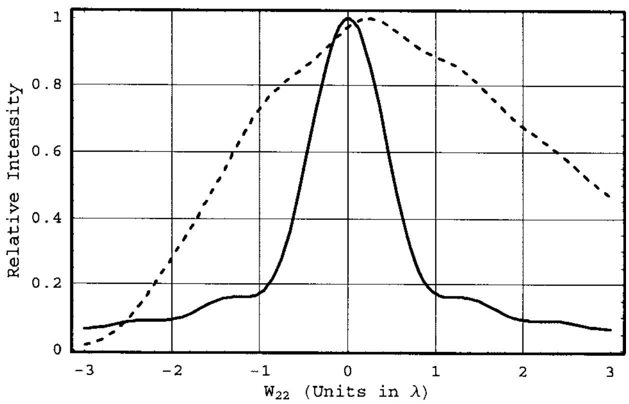

Fig. 4. Variation of the central intensity as a function of astigmatism W_{22} with a QF with $\alpha=15$ and $\alpha_0=9.75$ (dashed curve) and without a QF (solid curve). The normalization coefficient for the QF intensity is 0.046.

large negative W_{22}. On the other hand, a positive W_{22} tends to move the stationary point away from the boundary and diminish its effects on the intensity variation only affected now by astigmatism.

The performance of the QF in the presence of astigmatism is analyzed in the spatial-frequency domain by calculation of the MTFs for $0<W_{22}<3$ (see Fig. 2). In a similar fashion to the effect of coma on the MTF, the MTFs remain fairly invariant and the cutoff frequency has been reduced by a factor of 2, giving rise to broader point-spread functions, as shown in Fig. 3. In contrast to coma, it becomes apparent that the QF MTFs display superior performance over the MTFs obtained from stopping down the aperture, as there are no zeros and the MTF variation with astigmatism is attenuated.

4. CONCLUSION

The stationary phase method has been used to obtain a phase filter that, in principle, provides reduced sensitivity to coma and to astigmatism; in both cases the numerically calculated MTF remains almost constant as the parameter for either astigmatism or coma is varied. Although the invariance to astigmatism offers a means for useful enhancement in performance, for optical systems exhibiting coma useful invariance can also be achieved by a reduction in the aperture size to yield the same effective cutoff frequency, which negates the usefulness of the QF for this application. This work presented here complements work previously published[5,10] where QFs have been shown to enhance the tolerance to defocus and spherical aberration. It is notable that the effective cutoff frequency is also reduced, indicating that for applications employing pixelated detectors these filters may be usefully employed as antialiasing filters.

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